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On the uniqueness of the mixed equilibrium in the Tullock contest☆

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ABSTRACT

It is shown that the two-player Tullock contest admits precisely one equilibrium in randomized strategies.

1. Preliminaries

1.1. Introduction

In Tullock's (1980) model of political conflict, contestants compete in efforts to obtain a reward that cannot be easily allocated otherwise. Suppose there are two contestants, A and B, with respective valuations of winning given by $V_A \geq V_B > 0$. When contestants choose efforts $x_A \geq 0$ and $x_B \geq 0$, contestant i's payoff is given by

$$u_i(x_A, x_B) = \frac{x_i^R}{x_A^R + x_B^R} V_i - x_i,$$

where $R \geq 0$ is an exogenous parameter, and the ratio is interpreted as $\frac{1}{2}$ if otherwise undefined. While that model has been the workhorse of contest theory for several decades (Beviá and Corchón, 2024), the equilibrium analysis remained incomplete. Specifically, prior work did not address the question whether the equilibrium for $R > 2$ is unique. In this paper, we establish the uniqueness of the mixed equilibrium in the two-player Tullock contest.

1.2. Statement of the main result

Suppose that each contestant $i \in \{A, B\}$ chooses a probability distribution over the interval $[0, V_i]$.[1] Then, payoffs are bounded, and a *mixed-strategy Nash equilibrium (MSNE)* may be defined as usual (Dasgupta and Maskin, 1986). The main result of the present paper is the following.

Theorem 1. *The two-player Tullock contest has a unique MSNE, for any $V_A \geq V_B > 0$ and $R > 2$.*

Proof. See Section 3. □

There is a sense in which Theorem 1 completes the equilibrium analysis of the two-player case. Indeed, for $R \in [0, 1 + (V_B/V_A)^R]$, it was known that a unique equilibrium in pure strategies exists (Nti, 1999). Moreover, for $R \in (1+(V_B/V_A)^R, 2]$, a semi-mixed equilibrium, in which contestant A plays a pure strategy while contestant B randomizes between a positive and a zero effort, was known to exist (Wang, 2010). Finally, for $R \in [0, 2]$, there are no other MSNE, *i.e.*, the equilibrium is always unique (Ewerhart, 2017b; Feng and Lu, 2017). Thus, by covering the remaining case $R > 2$, Theorem 1 indeed rounds up the equilibrium analysis of the two-player Tullock contest.[2]

1.3. Summary of the proof

The proof of Theorem 1 makes use of sequence spaces, Cauchy matrices, and Dirichlet series. Background information on these mathematical tools is provided in an Appendix. The proof has two steps. First,

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[1] This assumption is reasonable because bids that exceed the contestant's valuation are strictly dominated by the zero bid.

[2] In particular, our conclusions carry over to variants of the Tullock contest in which players differ in marginal cost or ability rather than valuation. Similarly, the assumption of nonlinear returns may be replaced by nonlinear cost, using a well-known substitution argument (Szidarovszky and Okuguchi, 1997; Cornes and Hartley, 2005).

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we rewrite the condition of complete rent dissipation as an operator equation in the Banach space of absolutely summable sequences. The operator is represented by an infinite symmetric Cauchy matrix with finite supremum norm. Second, we show that the infinite Cauchy matrix is positive definite.[3] For this, we rewrite the infinite Cauchy matrix as an integral over a parameterized matrix that decomposes naturally as the outer product of two identical infinite vectors. This approach reduces the problem of uniqueness of the mixed equilibrium to the question if the coefficients of a Dirichlet series are uniquely determined. As the representation is indeed unique (Hardy and Riesz, 1915), this completes the proof.

1.4. Overview

The remainder of the paper is structured as follows. Section 2 reviews prior work. Section 3 presents the proof of Theorem 1. The Appendix provides the necessary background on the mathematical tools employed in the proof.

2. Prior work on the case $R > 2$

We are aware of four papers that made progress on the research question addressed in the present paper.

Assuming homogeneous valuations and $R > 2$, Baye et al. (1994) demonstrated that a MSNE with complete rent dissipation exists. In their proof, they first derived a bound on the equilibrium payoff in a discrete version of the game and then employed a limit argument. They also explained why the first-order conditions do not identify a symmetric Nash equilibrium in pure strategies, thereby resolving an important puzzle in the earlier literature.

Alcalde and Dahm (2010) defined an *all-pay auction equilibrium* in a probabilistic contest as a MSNE in which bids, winning probabilities, and payoffs coincide, in expectation, with the corresponding values for the all-pay auction (in the equilibrium with two active bidders). Moreover, they identified conditions under which a MSNE in a two-player contest with homogeneous valuations can be transformed into an all-pay auction equilibrium in a contest with heterogeneous valuations. The construction modifies the equilibrium strategy of one player by having her abstain from bidding with positive probability.[4]

In Ewerhart (2017a), the equilibrium set of probabilistic contests was characterized. In particular, it was shown that any MSNE of the two-player Tullock contest with $R > 2$ can be constructed as an all-pay auction equilibrium. Therefore, uniqueness of the MSNE in the special case where $V_A = V_B > 0$ implies uniqueness for general valuations $V_A \geq V_B > 0$. Furthermore, since a homogeneous prize in the Tullock contest can always be normalized to unity without loss of generality, it follows that verifying the uniqueness claim in Theorem 1 is sufficient in the special case where $V_A = V_B = 1$ and $R > 2$.

Focusing on this special case, Ewerhart (2015) further explored the nature of the MSNE. The following result will be utilized in the proof of Theorem 1.

Lemma 1 (*Ewerhart, 2015*). *Suppose that $V_A = V_B = 1$ and $R > 2$. Then, there is a strictly declining sequence of positive bid levels $\{y_k\}_{k=1}^{\infty}$ with $\lim_{k\to\infty} y_k = 0$ that jointly nest the support of any equilibrium strategy. Further, for any $k \in \{1, 2, \ldots\}$, we have the condition of complete rent dissipation at y_k,*

$$y_k = \sum_{l=1}^{\infty} \frac{q_l y_k^R}{y_k^R + y_l^R},$$

where q_l denotes the probability attached by the equilibrium strategy to the bid level y_l.

Proof. See Ewerhart (2015). □

Thus, under the assumptions of Lemma 1, any equilibrium strategy places all probability weight on countably many positive bid levels that are contained in a known set. In particular, any such strategy may be represented by a probability distribution $\{q_k\}_{k=1}^{\infty}$.

3. Proof of Theorem 1

Consider the infinite symmetric matrix

$$Y = \left\{ \frac{y_k^R y_l^R}{y_k^R + y_l^R} \right\}_{k,l=1}^{\infty},$$

where the sequence $\{y_k\}_{k=1}^{\infty}$ is defined via Lemma 1. Given that $\{y_k\}_{k=1}^{\infty}$ is strictly declining, the entries of Y are bounded by

$$\frac{y_k^R y_l^R}{y_k^R + y_l^R} \leq \frac{y_1^R}{2}.$$

The following lemma says that, if probability distributions $q^* = \{q_k^*\}_{k=1}^{\infty}$ and $q^{**} = \{q_k^{**}\}_{k=1}^{\infty}$ each represent an equilibrium strategy, then the difference $q^* - q^{**} \in \ell^1(\mathbb{R})$ is in the null space of Y.

Lemma 2. *Let $q^* = \{q_k^*\}_{k=1}^{\infty}$ and $q^{**} = \{q_k^{**}\}_{k=1}^{\infty}$ be probability distributions each representing an equilibrium. Then, for any $k \in \{1, 2, \ldots\}$, we have*

$$\sum_{l=1}^{\infty} \frac{y_k^R y_l^R}{y_k^R + y_l^R} (q_l^* - q_l^{**}) = 0.$$

Proof. From Lemma 1, we see that

$$y_k = \sum_{l=1}^{\infty} \frac{q_l^* y_k^R}{y_k^R + y_l^R}.$$

Hence, exploiting that q^* is a probability distribution,

$$1 - y_k = \sum_{l=1}^{\infty} q_l^* \left(1 - \frac{y_k^R}{y_k^R + y_l^R} \right) = \sum_{l=1}^{\infty} \frac{q_l^* y_l^R}{y_k^R + y_l^R}.$$

Multiplying through with y_k^R yields

$$\sum_{l=1}^{\infty} \frac{q_l^* y_k^R y_l^R}{y_k^R + y_l^R} = \left(1 - y_k\right) y_k^R.$$

Subtracting the analogous relationship in with q^* is replaced by q^{**} yields the claim. □

We note now that

$$Y = \left\{ \frac{1}{y_k^{-R} + y_l^{-R}} \right\}_{k,l=1}^{\infty}.$$

The following result, which might be of independent interest, therefore completes the proof of Theorem 1.

Lemma 3. *Let $C = \{1/(\lambda_k + \lambda_l)\}_{k,l=1}^{\infty}$ be an infinite symmetric Cauchy matrix with $0 < \lambda_1 < \lambda_2 < \cdots$ and $\lim_{k\to\infty} \lambda_k = \infty$. Then, $C\alpha = 0$ implies $\alpha = 0$, for any $\alpha \in \ell^1(\mathbb{R})$.*

Proof. We note that

$$C = \int_0^{\infty} B(s) ds,$$

where

$$B(s) = \{\exp(-s(\lambda_k + \lambda_l))\}_{k,l=1}^{\infty}.$$

[3] As detailed in the Appendix, the standard definition of positive definiteness for finite matrices (Debreu, 1952) extends naturally to this case.

[4] From Alcalde and Dahm (2010, proof of Thm. 3.2), we also deduce that the uniqueness assertion of Theorem 1 does not extend to $N \geq 3$ contestants. Specifically, with $R > 2$ and homogeneous valuations, multiple all-pay auction equilibria exist, depending on which two contestants are selected to be active.

Moreover, $B(s) = b(s)b(s)^T$, with

$$b(s) = \{\exp(-s\lambda_k)\}_{k=1}^{\infty} \in \ell^\infty(\mathbb{R}),$$

for any $s \geq 0$. Take now some $\alpha \in \ell^1(\mathbb{R})$. Then,

$$\begin{aligned} \alpha^T B(s)\alpha &= \sum_{k,l=1}^{\infty} \exp(-s(\lambda_k + \lambda_l))\alpha_k\alpha_l \\ &= \left(\sum_{k=1}^{\infty} \exp(-s\lambda_k)\alpha_k\right)^2 \\ &= \left|b(s)^T \alpha\right|^2. \end{aligned}$$

Noting that

$$b(s)^T \alpha = \sum_{k=1}^{\infty} \alpha_k \exp(-s\lambda_k) \leq \|\alpha\|_1 \exp(-s\lambda_1)$$

holds for any $s \geq 0$, Lebesgue's dominated convergence theorem implies

$$\int_0^\infty \left|b(s)^T \alpha\right|^2 ds = \alpha^T C\alpha.$$

Hence, if $C\alpha = 0$, then the Dirichlet series $\sum_{k=1}^{\infty} \alpha_k \exp(-s\lambda_k)$ vanishes almost everywhere on the positive real axis. By Lemma A.1 in the Appendix, this shows that the coefficients α_k all vanish. Hence, $\alpha = 0$, which proves the lemma. □

Appendix. Mathematical tools

This section provides background on the mathematical tools used in the proof of Theorem 1. Specifically, we discuss sequence spaces (Appendix A.1), Cauchy matrices (Appendix A.2), and Dirichlet series (Appendix A.3).

A.1. Sequence spaces

Let $\ell^1(\mathbb{R}) = \{x = (x_1, x_2, \ldots) : \|x\|_1 < \infty\}$ denote the Banach space of absolutely summable sequences, where $\|x\|_1 = \sum_{k=1}^{\infty} |x_k|$. Further, let $\ell^\infty(\mathbb{R}) = \{x = (x_1, x_2, \ldots) : \|x\|_\infty < \infty\}$ denote the Banach space of bounded sequences, where $\|x\|_\infty = \sup_{k\in\{1,2,\ldots\}} |x_k|$. Then, the product $x^T b \equiv b^T x \equiv \sum_{k=1}^{\infty} x_k b_k \in \mathbb{R}$ converges absolutely for any $x \in \ell^1(\mathbb{R})$ and $b \in \ell^\infty(\mathbb{R})$ (Aliprantis and Border, 1994, Ch. 16).

Let $\mathcal{M}^\infty(\mathbb{R})$ denote the space of infinite matrices $A = \left\{a_{k,l}\right\}_{k,l=1}^{\infty}$ (Cooke, 1950) with finite supremum norm, *i.e.*, matrices for which $\sup_{k,l} |a_{k,l}| < \infty$. Then, for any $A \in \mathcal{M}^\infty(\mathbb{R})$ and $x \in \ell^1(\mathbb{R})$, we may define $Ax \in \ell^\infty(\mathbb{R})$ component-wise via $(Ax)_k = \sum_{l=1}^{\infty} a_{k,l}x_l \in \mathbb{R}$. In particular, for any $x, \hat{x} \in \ell^1(\mathbb{R})$, we have $x^T A\hat{x} \equiv (Ax)^T \hat{x} = x^T(A\hat{x}) = \sum_{k,l=1}^{\infty} a_{k,l}x_k\hat{x}_l \in \mathbb{R}$. The *null space* of $A \in \mathcal{M}^\infty(\mathbb{R})$ is the set of $x \in \ell^1(\mathbb{R})$ such that $Ax = 0$.

An infinite matrix $A = \left\{a_{k,l}\right\}_{k,l=1}^{\infty}$ is called *symmetric* if $a_{k,l} = a_{l,k}$ for all $k, l \in \{1, 2, \ldots\}$. An example is the *outer product* $bb^T = \{b_k b_l\}_{k,l=1}^{\infty} \in \mathcal{M}^\infty(\mathbb{R})$, where $b \in \ell^\infty(\mathbb{R})$. We say that a symmetric infinite matrix $A \in \mathcal{M}^\infty(\mathbb{R})$ is *positive semi-definite* if $x^T Ax \geq 0$ holds for all $x \in \ell^1(\mathbb{R})$. If, in addition, $x^T Ax = 0$ implies $x = 0$, then we say that A is *positive definite* .

A.2. Cauchy matrices

Given positive parameters $c_1, \ldots, c_n$, for some finite $n \geq 1$, the matrix

$$C_n = \left\{\frac{1}{c_k + c_l}\right\}_{k,l=1}^{n}$$

is called a symmetric Cauchy matrix. If the parameters $c_1, \ldots, c_n$ are, in addition, pairwise distinct, then C_n is positive definite (Fiedler, 2010, Thm. A).

Infinite Cauchy matrices may be defined by letting $n = \infty$. As noted by Schur (1911, p. 18), infinite Cauchy matrices with pairwise different entries are positive semi-definite. Lemma 3 in the body of the paper provides conditions under which such matrices are positive definite.

A.3. Dirichlet series

An infinite series of the form

$$f(s) = \sum_{k=1}^{\infty} \alpha_k \exp(-s\lambda_k), \tag{A.1}$$

for a sequence of coefficients $\{\alpha_k\}_{k=1}^{\infty}$ and a *frequency* $\lambda = \{\lambda_k\}_{k=1}^{\infty}$, where $\lim_{k\to\infty} \lambda_k = \infty$, is called a *(general) Dirichlet series*. In the special case where $\lambda_k = k$, the series in (A.1) reduces to a power series in the variable $z = \exp(-s)$. Our interest, however, lies in the case of a general frequency.

Lemma A.1 (*Hardy and Riesz, 1915*). *Suppose that the series in (A.1) is convergent for $s = 0$, and that for some $\delta > 0$, we have $f(s) = 0$ for infinitely many $s \geq \delta$. Then $\alpha_k = 0$ for all $k \in \{1, 2, \ldots\}$.*

Proof. See Hardy and Riesz (1915, Thm. 6). □

Data availability

No data was used for the research described in the article.

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